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1988 J. Phys. A: Math. Gen. 21 L923

(http://iopscience.iop.org/0305-4470/21/19/001)

J. Phys. A: Math. Gen. 21 (1988) L923-L927. Printed in the UK

LETTER TO THE EDITOR

Logarithmic strange sets

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Received 7 June 1988

Abstract. We study a simple model of a hierarchical fractal set exhibiting a thermodynamical behaviour which is similar to the one numerically observed in the analysis of the critical line for cricle maps: in particular, the free energy shows a phase transition whose nature depends on the parameters of the model.

Extensive studies on the thermodynamical behaviour of the set of irrational windings in critical circle maps [1] revealed that finite-size logarithmic effects play an important role; if one considers, for instance, the Farey tree partitioning of rationals [1, 2] and uses uniform weights at each level of the hierarchical construction of the set, the free energy exhibits a phase transition, whose details can hardly be investigated numerically, due to extremely slow convergence. This phase transition is indeed relevant, as it resides on the Hausforff dimension, which is conjectured to be universal for critical circle maps [3].

Our purpose is to introduce a prototype model of a logarithmic stange set in which the asymptotic thermodynamics can be analysed in some detail: again we will observe a phase transition at Hausdorff dimension, and if we tune the parameters to mimic critical circle map behaviour we find that the transition is continuous.

The strange set we consider is a sort of binary Cantor set in which the lengths can decrease either exponentially or in a polynomial way. Its corresponding Ising model [4] would be similar to the Fisher model [5], which was used by Katzen and Procaccia [6] to provide a tentative explanation of the features of the phase transition exhibited by the Julia set of a particular quadratic mapping.

The hierarchical construction of the set proceeds as follows. The covering intervals at the nth step have lengths that depend on the whole binary history:

$$l_{(n)}(k_1, \ldots, k_j) = \alpha^j \prod_{i=1}^{j} k_i^{-\gamma}$$

where k_i is the cardinality of identical bit substrings in the symbolic sequence: we obviously have $\Sigma_i\, k_i = n$. For instance, at level 5, the scale associated with 01100 is

$$l_5(1, 2, 2) = \alpha^3 \frac{1}{2^{2\gamma}}.$$

Obvious restrictions have to be put on the coefficients in order to prevent overlaps at each level. The geometric scale is associated with an alternating symbolic sequence

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0305-4470/88/190923+05$02.50

($0101\ldots01 \mapsto \alpha^n$), while repetitions of the same bit give $l = n^{-\gamma}$. We recall that this is what happens if we organise mode-locked intervals along the critical line down a Farey tree: the alternating sequence corresponds to the golden mean approximants, while the single-bit repetition gives the harmonic sequence. More generally, every time we have sequences of three different bits in the symbolic sequence the denominator of the third Farey fraction is given recursively from the preceding denominators via a Fibonacci-like relation. As the logarithm of the scale can be viewed as the Hamiltonian of the associated Ising system [4], we see that here a straight line down the binary tree corresponds to a cluster with a logarithmic surface term in the Fisher model.

We will perform our calculations using the grand canonical formalism [7], as this is the optimal choice for the combinatorics of the problem. We also remark that, in a somehow similar formalism, Fisher's model was used [8] to show that Ruelle's zeta function [9] cannot always be extended to a meromorphic function in the whole complex plane for a flow, even if the same satisfies axiom A.

The grand canonical partition function $\mathscr{Z}(z, \beta)$ [6] is

$$\mathscr{Z}(z, \beta) = \sum_{n=1}^{\infty} z^n Z_n(\beta)$$

where

$$Z_n(\beta) = \sum_{i=1}^{2^n} l_i^{\beta}.$$

Rearranging the order of the sums we get

$$\mathscr{Z}(z, \beta) = \sum_{n=1}^{\infty} z^n\, 2 \sum_{l=1}^{n} \sum_{m_1+\ldots+m_l=n} \alpha^{l\beta} m_1^{-\gamma\beta} \ldots m_l^{-\gamma\beta} = 2 \sum_{n=1}^{\infty} \left(\sum_{l=1}^{\infty} l^{-\gamma\beta_z l} \right)^n \alpha^{n\beta}$$

i.e. a geometric series. The asymptotic free energy is

$$F(\beta) = -\log(\hat{z}(\beta))/\log 2$$

where $\hat{z}(\beta)$ is the smallest z that makes $\mathscr{Z}$ divergent. Now, analogously to the Fisher model, we have two conditions that have to be satisfied in order that this expression converge: one internal condition

$$z < 1$$

and one external condition

$$\Phi(\beta\gamma, z) = \sum_{l=1}^{\infty} l^{-\beta\gamma_z l} < \alpha^{-\beta}$$

so that the free energy is determined by the smallest z violating either of these conditions.

We observe that there is a critical value β^* such that, for $\beta \geqslant \beta^*$, z is determined by the internal condition (fluid phase), while for smaller values of β z is implicitly given by the external condition. As a matter of fact, if $\beta > \beta^*$, where

$$\zeta(\gamma\beta^*) = \alpha^{-\beta^*} \tag{1}$$

(assuming $\gamma > 0$ and $\alpha < 1$ it is easy to see that there is always one, and only one, $\beta^* > 1/\gamma$ satisfying (1)) we have that, for $z = 1$, the external condition is still satisfied.

The behaviour of Φ and its derivatives, which determine the order of the phase transition, can be studied either by considering the original series or by observing that

$$\Phi(\mu, z) = \frac{1}{\Gamma(\mu)} \int_0^{\infty} \frac{z x^{\mu-1}\, \mathrm{d}x}{\mathrm{e}^x - z}$$

when $|z|<1$ and Re $\mu>0$. We have to consider the derivatives of $\hat{z}(\beta)$, defined by

$$\Theta(\beta\gamma, \hat{z}(\beta)) = \Phi(\beta\gamma, \hat{z}(\beta)) - \alpha^{-\beta} = 0 \qquad \beta \leqslant \beta^* \qquad \hat{z} \leqslant 1$$

with respect to β, and their limiting behaviour as $\beta \mapsto \beta^*$ ($\hat{z} \mapsto 1$). In particular the first deriviative is

$$\hat{z}'(\beta)|_{\beta=\beta^*} = -\frac{\partial_\beta \Theta(\beta\gamma, z)}{\partial_z \Theta(\beta\gamma, z)}\bigg|_{z\mapsto 1, \beta \mapsto \beta^*}. \tag{2}$$

While the numerator converges, what happens to the denominator is determined by the value of $\beta^*\gamma$: if $\beta^*\gamma$ (which is surely bigger than one) is less than two then

$$\partial_z \Phi(\beta^*\gamma, 1) = \frac{1}{\Gamma(\beta^*\gamma)} \int_0^\infty \frac{x^{\beta^*\gamma - 1}\, e^x \, dx}{(e^x - 1)^2}$$

diverges, and all derivatives are continuous through the transition point. Conversely, provided that $\beta^*\gamma > 2$, (2) tends to a finite limit, thus determining the first-order nature of the transition.

It is interesting to note that, in this case too, as in related models studied in [1], the phase transition point corresponds to the Hausdorff dimension, as the fluid phase has a vanishing free energy. To give a pictorial illustration of the resulting thermodynamical functions we have selected [1, 2, 10] $\gamma = 3$ and $\alpha = [(\sqrt{5}-1)/2]^{2.16443}$, which reproduce the extremal harmonic and geometric behaviour of critical circle maps: the Hausdorff dimension turns out to be 0.602 204, so that $\gamma\beta^* < 2$ and thus the thermodynamical functions are continuous along the critical line: in figures 1 and 2 the free energy and its first derivative are plotted.

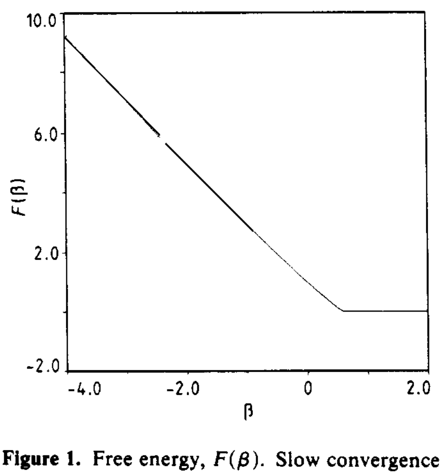

Figure 1. Free energy, $F(\beta)$. Slow convergence of the series in the proximity of the phase transition requires the use of logarithmic convergence acceleration; we used Levin's extrapolation [11].

We point out that this kind of phase transition is induced by 'pure' polynomial lengths: if such scalings just modulate an overall geometric scale the phase transition is moved to infinity. Consider for example

$$l_{(n)}(k_1, \ldots, k_j) = \alpha^n \prod_{i=1}^{j} k_i^{-\gamma}.$$

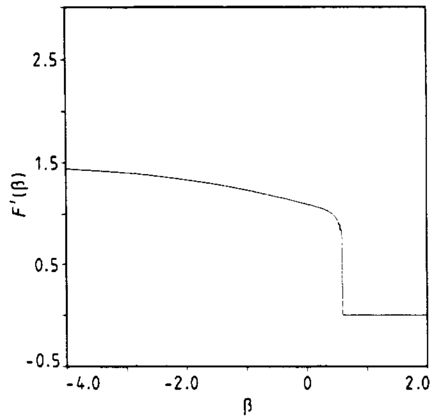

Figure 2. First derivative of the free energy, $F'(\beta)$.

These scalings lead to the grand canonical partition function

$$\mathscr{Z}(\beta, z) = \sum_{n=1}^{\infty} \left(\sum_{l=1}^{\infty} \alpha^{l\beta} z^{l} \frac{1}{l^{\gamma\beta}} \right)^{n}.$$

As $\zeta(\sigma) > 1$ for all σ the internal condition never dominates and we never observe any phase transition.

The importance of pure polynomial terms (i.e. contributions slowly converging to one in the scaling function) can be seen also by considering a one-scale Cantor set in which, at each level, only the endpoint intervals scale with a power law [1]. The nth level canonical partition function is

$$Z_n(\beta) = 2^{n-1}\alpha^{n\beta} + 2/n^{\beta\gamma},$$

which leads to

$$\mathscr{Z}(z, \beta) = 2\Phi(\beta\gamma, z) + \frac{1}{2}\frac{1}{1-2\alpha^{\beta}z}.$$

Even in this case we have two different conditions for convergence: $z < 1$ (first, 'infinite cluster' term) and $z < \frac{1}{2}\alpha^{-\beta}$. The transition point $\beta^* = \log 2/\log \alpha$ is again the Hausdorff dimension (independent of γ in this case and coincident with the one for the simple one-scale Cantor set). The nature of the phase transition also depends in this case upon γ.

I thank Predrag Cvitanović for many discussions and suggestions. This work was partially supported by Fondazione Angelo Della Riccia.

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